Matrices

Dr Gavin M Abernethy

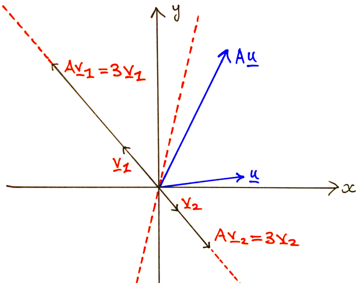
Lecture 5

Today we shall cover:

- How to calculate eigenvalues and eigenvectors by hand for 2×2 matrices.
- Using MATLAB to obtain eigenvalues and eigenvectors for 3×3 and larger matrices.

Revision: What are eigenvalues and eigenvectors?

When a 2×2 matrix A pre-multiplies a position vector, it is usually stretched and rotated. However, there exist "natural axes" of vectors (the eigenvectors) whose direction stays the same, and they are simply scaled by a constant (the eigenvalue).



$\underline{\mathbf{u}}$ is *not* an eigenvector of A, but both $\underline{\mathbf{v}}_1$ and $\underline{\mathbf{v}}_2$ *are*, with an eigenvalue of 3. In fact, *all* vectors on the red axes are eigenvectors.

How to calculate eigenvalues and eigenvectors (1)

To find the eigenvalues λ and eigenvectors $\underline{\mathbf{x}}$ of matrix A, we first re-arrange the definition $A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}}$ to:

$$(A - \lambda I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

where I is the identity matrix.

Then to find the eigenvalues, solve the following equation for λ:

Characteristic polynomial of A

$$det\left(A - \lambda I\right) = 0$$

For a 2×2 matrix this will be a **quadratic equation**.

How to calculate eigenvalues and eigenvectors (2)

Then each eigenvalue $\lambda = \lambda_1, \lambda_2, \ldots,$ we then obtain a corresponding non-zero eigenvector $\underline{\mathbf{x}} = \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \ldots$

We can do this by substituting in the eigenvalue and solving:

To find the eigenvector:

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \qquad \text{or} \qquad (A - \lambda I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

for the column vector $\underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{pmatrix}$

How to calculate eigenvectors

Recall from last week, that there are **infinitely-many** eigenvectors corresponding to each eigenvalue.

This means that when solving the set of equations to find $\underline{\mathbf{x}}$, there is *no single solution*. Instead, we can choose a value for one of the variables, and then use the equations to obtain the remainder.

This also results in redundancy among the equations.
In the 2×2 case, the two equations obtained will be the **same**.

Example: Eigenvalues and Eigenvectors of a 2×2 Matrix

Determine the eigenvalues and eigenvectors of the following 2×2 matrix:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix}$$

Example: Eigenvalues and Eigenvectors of a 2×2 Matrix

First determine the eigenvalues:

$$\begin{aligned} A - \lambda I &= \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \\ &= \begin{pmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{pmatrix} \end{aligned}$$

Therefore, we wish to solve the characteristic equation:

$$|A - \lambda I| = 0 \quad \Longrightarrow \quad \begin{vmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{vmatrix} = 0$$

Example: Eigenvalues and Eigenvectors of a 2×2 Matrix

Evaluating the determinant by taking the difference of the product of the diagonals:

$$(1-\lambda)(-4-\lambda)-(2)(3)=0,$$

and so the characteristic polynomial is:

$$\lambda^2+3\lambda-10=0$$

Solving this quadratic equation yields two distinct, real, integer roots:

$$\lambda_1=-5, \qquad \lambda_2=2$$

These are the two eigenvalues of A.

Example: Eigenvalues and Eigenvectors of a 2×2 Matrix

Next we solve the eigenvectors one at a time.

For the first eigenvalue, $\lambda_1 = -5$, call the corresponding eigenvector $\underline{\mathbf{e}}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$

To find the values of the components x and y, we need to solve:

$$A\underline{\mathbf{e}}_1 = -5\underline{\mathbf{e}}_1$$

$$\therefore \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -5 \begin{pmatrix} x \\ y \end{pmatrix}$$

The rows of this matrix equation yield a pair of equations.

Example: Eigenvalues and Eigenvectors of a 2×2 Matrix

$$\begin{aligned} x + 2y &= -5x \\ 3x - 4y &= -5y \end{aligned}$$

These two are actually the same equation, rearranged.

Solving either yields:

$$y = -3x$$

If we choose $x = 1$ (since *any* scalar multiple of the eigenvector will work), then $y = -3$.

So one eigenvector corresponding to $\lambda_1 = -5$ is:

$$\underline{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Example: Eigenvalues and Eigenvectors of a 2×2 Matrix

Using the same method, can you find the second eigenvector (corresponding to eigenvalue $\lambda_2 = 2$)?

Example: Eigenvalues and Eigenvectors of a 2×2 Matrix

Call the corresponding eigenvector $\underline{\mathbf{e}}_2 = \begin{pmatrix} x \\ y \end{pmatrix}$

To find the values of the components x and y, we need to solve:

$$A\underline{\mathbf{e}}_2 = 2\underline{\mathbf{e}}_2$$

$$\therefore \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = 2 \begin{pmatrix} x \\ y \end{pmatrix}$$

The rows of this matrix equation yield a pair of equations:

$$\begin{aligned} x + 2y &= 2x \\ 3x - 4y &= 2y \end{aligned}$$

Example: Eigenvalues and Eigenvectors of a 2×2 Matrix

Again these are actually the same equation. Can you see why?

We can rearrange either to:

$$-x + 2y = 0$$

and so

$$x = 2y$$

Choose $y = 1$, then it follows that $x = 2$ and so:

$$\underline{\mathbf{e}}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

MATLAB

We can determine the eigenvalues and eigenvectors of a square matrix A using MATLAB. This can be used for solving 2×2 matrices, but also 3×3, 4×4 or even larger matrices.

```
A = [1 2; 3 -4];

B = sym(A);

[vecA,valA] = eig(B);
```

This first creates a symbolic version B and then produces two matrices: *valA* contains the eigenvalues on the diagonal (with zeros elsewhere) and *vecA* contains the eigenvectors in each column. The ordering corresponds, so the first column of *vecA* is the eigenvector corresponding to the eigenvalue in the first diagonal entry of *valA*.

MATLAB

To find a unit vector, we need to first obtain the magnitude of the vector, and then divide the vector by this magnitude:

`X = [3; 17];`	(declare a vector X)
`mag = norm(X);`	(obtain the magnitude of X)
`unitX = X/mag;`	(obtain the unit vector)

In most of the examples of $n \times n$ matrices that we will consider, there will be exactly n real eigenvalues. However,

- It is possible that the eigenvalues will be **complex numbers** (in the form $a + bj$ where $j = \sqrt{-1}$ is the imaginary number).
- Some of the eigenvalues may **repeat**. In this case, we can separate the eigenvector solution into a combination of two independent eigenvectors, so that we end up with n eigenvectors in total.

We shall see worked examples of both in the tutorials.

Exercises

Obtain the eigenvalues and corresponding eigenvectors of the following square matrices, and in each case illustrate the eigenvectors on a graph:

1. $C = \begin{pmatrix} -4 & -2 \\ 11 & 9 \end{pmatrix}$
2. $D = \begin{pmatrix} 5 & 3 \\ 6 & 2 \end{pmatrix}$

Solutions

1. $C = \begin{pmatrix} -4 & -2 \\ 11 & 9 \end{pmatrix}$

   $$\lambda_1 = 7, \;\; \lambda_2 = -2$$

   $$\underline{\mathbf{e}}_1 = \begin{pmatrix} 2 \\ -11 \end{pmatrix}, \;\; \underline{\mathbf{e}}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2. $D = \begin{pmatrix} 5 & 3 \\ 6 & 2 \end{pmatrix}$

   $$\lambda_1 = 8, \;\; \lambda_2 = -1$$

   $$\underline{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \;\; \underline{\mathbf{e}}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$