Matrices

Dr Gavin M Abernethy

Lecture 6

Today we shall cover:

- How to calculate eigenvalues and eigenvectors by hand for 3×3 matrices.
- This follows the same procedure as for 2×2 matrices, but we need to remember how to take the determinant of a 3×3 matrix.
- We shall also apply our learning to the stability applications introduced in Lecture 4.

Revision: How to calculate eigenvalues and eigenvectors

To find the eigenvalues λ and eigenvectors $\underline{\mathbf{x}}$ of matrix A, we first re-arrange the definition $A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}}$ to:

$$(A - \lambda I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

where I is the identity matrix.

Then to find the eigenvalues, solve the following equation for λ:

Characteristic polynomial of A

$$\det\left(A - \lambda I\right) = 0$$

Revision: How to calculate eigenvalues and eigenvectors

For each eigenvalue $\lambda = \lambda_1, \lambda_2, \ldots$, we then obtain a corresponding non-zero eigenvector $\underline{\mathbf{x}} = \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2, \ldots$

We can do this by substituting in the eigenvalue and solving:

To find the eigenvector:

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \qquad \text{or} \qquad (A - \lambda I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

for the column vector $\underline{\mathbf{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$ in the 3×3 case.

Example: Eigenvalues and Eigenvectors of a 3×3 Matrix

Consider the following 3×3 matrix A.

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

We will calculate the three eigenvalues and associated eigenvectors for this matrix.

Example: Eigenvalues and Eigenvectors of a 3×3 Matrix

$$|A - \lambda I| = 0 \quad \text{gives} \quad \begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} -1 & 2-\lambda \\ 0 & -1 \end{vmatrix} = 0$$

$$\therefore (1-\lambda)\big((2-\lambda)(1-\lambda) - (-1)(-1)\big) + \big((-1)(1-\lambda) - (-1)(0)\big)$$
$$+0\big((-1)(-1) - (2-\lambda)(0)\big) = 0$$

This reduces to:

$$(1-\lambda)(\lambda)(\lambda-3) = 0$$

Hence there are three eigenvalues: $\lambda = 0, 1, 3$.

Example: Eigenvalues and Eigenvectors of a 3×3 Matrix

Note: In this case, we kept out the common factor of $(\lambda - 1)$. You could instead be given the eigenvalues and asked to **verify** them, meaning that you must obtain the characteristic polynomial, then show by substitution that the proposed value satisfies the equation.

e.g. If we had multiplied out the characteristic polynomial to obtain:

$$\lambda^3 - 4\lambda^2 + 3\lambda = 0$$

Then to verify that $\lambda = 3$ is an eigenvalue:

$$\begin{aligned} (3)^3 - 4(3)^2 + 3(3) &= 27 - 4 \times 9 + 9 \\ &= 27 - 36 + 9 \\ &= 0 \end{aligned}$$

Example: Eigenvalues and Eigenvectors of a 3×3 Matrix

i) For the first eigenvalue $\lambda_1 = 0$, let $\underline{\mathbf{e}}_1 = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$ be the corresponding eigenvector.

$$A\underline{\mathbf{e}}_1 = \lambda_1 \underline{\mathbf{e}}_1 \implies \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 0 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

Multiplying out the rows gives three equations:

$$\begin{aligned} x - y \quad\quad &= 0 \\ -x + 2y - z &= 0 \\ - y \ + z &= 0 \end{aligned}$$

Example: Eigenvalues and Eigenvectors of a 3×3 Matrix

From the first equation, we obtain:

$$x - y = 0 \quad \Longrightarrow \quad x = y$$

From the third equation:

$$-y + z = 0 \quad \Longrightarrow \quad z = y$$

Thus, if we choose $y = 1$, then it follows that $x = 1$ and $z = 1$. As in the previous example, the final equation is redundant - but we should check by substitution that it agrees with our solution:

$$\begin{aligned} -x + 2y - z &= -(1) + 2(1) - (1) \\ &= -1 + 2 - 1 \\ &= 0 \quad \text{as expected!} \end{aligned}$$

Example: Eigenvalues and Eigenvectors of a 3×3 Matrix

Hence one eigenvector is:

$$\underline{\mathbf{e}}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Any other vector with the same direction (same ratio between the components) would *also* be an eigenvector for $\lambda_1 = 0$.

For example:

$$\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}, \quad \begin{pmatrix} -31 \\ -31 \\ -31 \end{pmatrix}, \quad \begin{pmatrix} 2.007 \\ 2.007 \\ 2.007 \end{pmatrix}$$

Or we could say: $\alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \forall \alpha \in \mathbb{R}$

to describe the set of all of them!

Example: Eigenvalues and Eigenvectors of a 3 × 3 Matrix

ii) $\lambda_2 = 1$:

$$A\underline{\mathbf{e}}_2 = \lambda_2 \underline{\mathbf{e}}_2 \implies \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 1 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This gives three equations:

$$\begin{aligned} x - y \qquad &= x \\ -x + 2y - z &= y \\ - y + z &= z \end{aligned}$$

Example: Eigenvalues and Eigenvectors of a 3 × 3 Matrix

From the first equation $x - y = x$, we have:

$$y = 0$$

Substituting this into $-x + 2y - z = y$ gives:

$$-x - z = 0 \qquad \Longrightarrow \qquad z = -x$$

Choose $x = 1$, then $y = 0$ and $z = -1$, so an eigenvector is:

$$\underline{\mathbf{e}}_2 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

iii) **As an exercise now**

Can you determine an eigenvector $\underline{\mathbf{e}}_3$ corresponding to the third eigenvalue:

$$\lambda_3 = 3$$

(Don't worry if you get a different answer from your neighbours - ask, is the *direction* the same?)

Example: Eigenvalues and Eigenvectors of a 3×3 Matrix

iii) $\lambda_3 = 3$:

$$A\underline{\mathbf{e}}_3 = \lambda_3 \underline{\mathbf{e}}_3 \implies \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = 3 \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

This gives three equations:

$$\begin{aligned} x - y \qquad &= 3x \\ -x + 2y - z &= 3y \\ -y + z &= 3z \end{aligned}$$

Example: Eigenvalues and Eigenvectors of a 3 × 3 Matrix

Simplifying,

$$\begin{aligned} -2x - y \quad &= 0 \\ -x - y - z &= 0 \\ - y - 2z &= 0 \end{aligned}$$

From the first equation:

$$y = -2x$$

and from the third equation:

$$z = -\frac{1}{2}y$$

So choose $x = 1$, then $y = -2$ and then $z = 1$. Hence,

$$\underline{\mathbf{e}}_3 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

Example: Eigenvalues and Eigenvectors of a 3×3 Matrix

The complete solution to the problem is therefore:

$$\lambda_1 = 0, \qquad \underline{\mathbf{e}}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$\lambda_2 = 1, \qquad \underline{\mathbf{e}}_2 = \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$$\lambda_3 = 3, \qquad \underline{\mathbf{e}}_3 = \gamma \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

for *any* real values of the scalar constants α, β, γ.

Return to the linear stability application

In Lecture 4, we briefly discussed an application: using eigenvalues to determine the stability of equilibria of systems of ODEs.

If we have a system of **linear** ODEs of the form:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = ax_1 + bx_2 + cx_3$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = dx_1 + ex_2 + fx_3$$
$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = gx_1 + hx_2 + ix_3$$

we can write it in matrix form as:

$$\dot{X} = AX$$

Return to the linear stability application

Where:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \dot{X} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}$$

and

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

is the **Jacobian matrix**.

(The dot above a variable denotes the derivative w.r.t. time)

Return to the linear stability application

Such a system has an **equilibrium** at the origin (where $x_1 = x_2 = x_3 = 0$), as this means that:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{\mathrm{d}x_3}{\mathrm{d}t} = 0$$

so every variable has zero rate of change with respect to time - i.e. the system is static. More complex non-linear systems may have other equilibria, or lack one at the origin, but we will only deal with this case.

A common question in the design of control systems, is when is this equilibrium **stable**?

Return to the linear stability application

Stability means that if the system is disturbed from the equilibrium by an arbitrarily small amount, it will tend to return (small perturbations dissipate). In an unstable system, the disturbance is amplified and over time the system will move away from the equilibrium. If this is desirable depends on the context - consider chemical reactions in an industrial process, some you want to sustain but others could be dangerous.

Returning to eigenvalues then. . .

Stability criterion:

The equilibrium of such a linear ODE system is:

- *Stable* if all of the eigenvalues of the Jacobian matrix have negative real part.
- *Unstable* otherwise.

Return to the linear stability application - Example (I/II)

Consider a process governed by the differential equations:

$$\dot{x} = x - y$$
$$\dot{y} = -x + 2y - z$$
$$\dot{z} = -y + z$$

Writing this in matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 1x - 1y + 0z \\ -1x + 2y - 1z \\ 0x - 1y + 1z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

we can see that this system has Jacobian equal to matrix A that we have studied today.

Return to the linear stability application - Example (II/II)

We check that there exists an equilibrium at the origin. If $x = y = z = 0$, then clearly from the original set of equations:

$$\dot{x} = \dot{y} = \dot{z} = 0 \quad \text{hence, equilibrium.}$$

(However, note that in this system, this zero-equilibrium is actually part of a family of equilibria for *any* case where $x = y = z$. Can you see why this is the case?)

Now, earlier we found that A had eigenvalues $\lambda = 0, 1, 3$. Thus, according to the criterion, the presence of positive, real eigenvalues means that this equilibrium is **unstable** and the system will tend to be repelled from it.

Onwards

I hope you have enjoyed the first three-quarters of the module.

Enjoy the rest of *Maths for Materials and Design*!