Maths for Materials and Design

Matrix Algebra

Contents

1 Introduction to Matrices

1.1 Definitions and Order

- **Definition:** A matrix is a rectangular array of numbers or letters. We use either square or round brackets and a capital letter to denote them. The ordering of elements matters.
- A vector is a matrix with only one column (i.e. a one-dimensional array of data), or only one row (a "row vector").
- **Order of a matrix:** the size and shape, described by the number of rows and then the number of columns.

Example 1.1.

$$\begin{pmatrix} 3 & 0 \\ 1 & -2 \\ -4 & 5 \end{pmatrix} \quad \textit{has order } 3 \times 2$$

$$\begin{pmatrix} 2 \\ -5 \end{pmatrix} \qquad \textit{This is a } 2 \times 1 \textit{ matrix. It is also a } \text{vector}.$$

$$\begin{pmatrix} 1 & -2 & 8 \\ 3 & 1 & 4 \end{pmatrix} \qquad \textit{This is a } 2 \times 3 \textit{ matrix.}$$

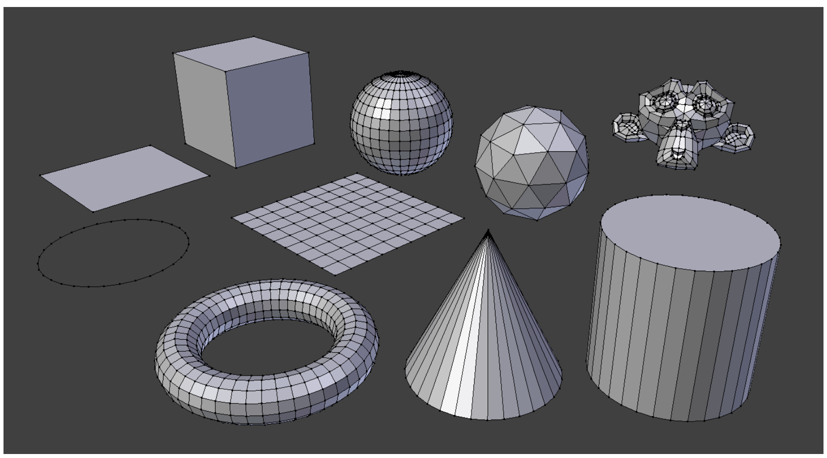
$$\begin{pmatrix} 2 & 0 & -1 & 6 \end{pmatrix} \qquad \textit{This is a } 1 \times 4 \textit{ matrix. It could be considered a "row vector".}$$

$$\begin{pmatrix} 10 & -2 & 8 \\ 3 & 1 & -9 \\ 11 & -2 & 7 \end{pmatrix} \qquad \textit{This is a } 3 \times 3 \textit{ matrix.}$$

1.2 Example application: Matrices as linear transformations

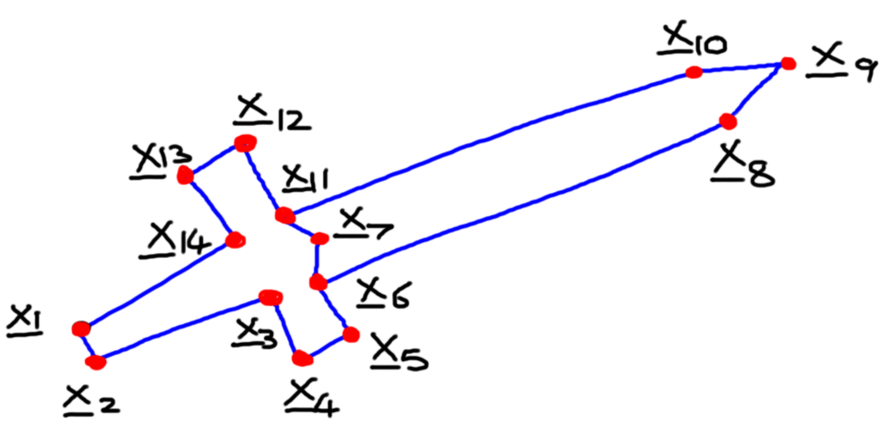
One application of matrix algebra is to 3d graphics engines, such as computer aided design (CAD), or how a video game engine understands the position and movement of objects. We will use this setting to help us think about the effect of the various matrix operations that we will be learning how to conduct.

For example, a game understands the position of a sword through vectors $\underline{\mathbf{x}}$ containing the initial co-ordinates of its vertices (corners).



Let's say that you have a set of n vectors $\underline{\mathbf{x}}_1, \ldots, \underline{\mathbf{x}}_n$ containing the co-ordinates of the vertices of the two-dimensional image of the sword:

$$\underline{\mathbf{x}}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix}$$



Then we can change how the image of the sword appears by applying various matrix operations to its set of vertices:

- To **translate** the image of the sword (to shift it up, down, left or right) we *add* a vector to each vertex.

  For example, adding the vector $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ would move the image 1 unit to the right and 2 units down.

- To **stretch** the image in all directions (i.e. to resize it without distortion), we can multiply the vertices by a scaling factor α. This is called *scalar multiplication*.

The action of swinging the sword to a new position may change the orientation of the image, rotating, distorting and stretching it in more complicated ways. This action can be encoded in a matrix A, and applying the matrix multiplication to all the position vectors associated with the sword is how the game figures out where it moves to.

This transformation can be expressed in matrix form as:

$$\underline{\mathbf{y}} = A\underline{\mathbf{x}}$$

Matrix multiplication determines the new position $\underline{\mathbf{y}}$.

1.3 Addition and Subtraction

In order to add or subtract two matrices, they *must* have exactly the same order (both the same number of rows, and the same number of columns). In this case we add or subtract each of their corresponding entries:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} a+e & b+f \\ c+g & f+h \end{pmatrix}$$

Example 1.2.

$$\begin{pmatrix} 3 & 0 \\ 1 & -2 \\ -4 & 5 \end{pmatrix} + \begin{pmatrix} -1 & -2 \\ 8 & 4 \\ 7 & 2 \end{pmatrix} = \begin{pmatrix} 3-1 & 0-2 \\ 1+8 & -2+4 \\ -4+7 & 5+2 \end{pmatrix} = \begin{pmatrix} 2 & -2 \\ 9 & 2 \\ 3 & 7 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -3 \\ 16 \end{pmatrix} + \begin{pmatrix} -4 \\ 7 \\ 0 \end{pmatrix} = \begin{pmatrix} 1-4 \\ -3+7 \\ 16+0 \end{pmatrix} = \begin{pmatrix} -3 \\ 4 \\ 16 \end{pmatrix}$$

$$\begin{pmatrix} 1 \\ -3 \\ 16 \end{pmatrix} + \begin{pmatrix} 12 & 4 \\ -5 & 2 \end{pmatrix} \quad \textit{Not valid as these matrices do not have the same order.}$$

Example 1.3. *Given that:*

$$A = \begin{pmatrix} 1 & 2 \\ -6 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & -2 \\ 5 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 7 & 1 & -3 \\ 5 & 3 & -1 \end{pmatrix}$$

Calculate the following, if they exist:

$$A + B, \qquad A + C, \qquad B - A$$

Solution:

$$A + B = \begin{pmatrix} 1 & 2 \\ -6 & 3 \end{pmatrix} + \begin{pmatrix} 4 & -2 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} 1+4 & 2-2 \\ -6+5 & 3+1 \end{pmatrix} = \begin{pmatrix} 5 & 0 \\ -1 & 4 \end{pmatrix}$$

$A + C$ *does not exist as they do not have the exact same order.*

$$B - A = \begin{pmatrix} 4 & -2 \\ 5 & 1 \end{pmatrix} - \begin{pmatrix} 1 & 2 \\ -6 & 3 \end{pmatrix} = \begin{pmatrix} 4-1 & -2-2 \\ 5-(-6) & 1-3 \end{pmatrix} = \begin{pmatrix} 3 & -4 \\ 11 & -2 \end{pmatrix}$$

1.4 Scalar Multiplication

To multiply a matrix by a scalar (a real or complex *number*, rather than a vector or matrix) we simply multiply ("scale") each element of the matrix by that scalar:

$$\alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix}$$

Example 1.4.

$$4 \begin{pmatrix} 3 & 0 \\ 1 & -2 \\ -4 & 5 \end{pmatrix} = \begin{pmatrix} 4 \times 3 & 4 \times 0 \\ 4 \times 1 & 4 \times -2 \\ 4 \times -4 & 4 \times 5 \end{pmatrix} = \begin{pmatrix} 12 & 0 \\ 4 & -8 \\ -16 & 20 \end{pmatrix}$$

$$7 \begin{pmatrix} 1 & -3 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 7 & -21 \\ 0 & 14 \end{pmatrix}$$

Example 1.5. *Given that:*

$$A = \begin{pmatrix} 1 & 2 \\ -6 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & -2 \\ 5 & 1 \end{pmatrix}, \qquad C = \begin{pmatrix} 7 & 1 & -3 \\ 5 & 3 & -1 \end{pmatrix}$$

Calculate the following:

$$3A, \qquad -2B, \qquad 5C, \qquad 3A - 2B$$

Solution:

$$3A = 3\begin{pmatrix} 1 & 2 \\ -6 & 3 \end{pmatrix} = \begin{pmatrix} 3 \times 1 & 3 \times 2 \\ 3 \times -6 & 3 \times 3 \end{pmatrix} = \begin{pmatrix} 3 & 6 \\ -18 & 9 \end{pmatrix}$$

$$-2B = -2\begin{pmatrix} 4 & -2 \\ 5 & 1 \end{pmatrix} = \begin{pmatrix} -2 \times 4 & -2 \times -2 \\ -2 \times 5 & -2 \times 1 \end{pmatrix} = \begin{pmatrix} -8 & 4 \\ -10 & -2 \end{pmatrix}$$

$$5C = 5\begin{pmatrix} 7 & 1 & -3 \\ 5 & 3 & -1 \end{pmatrix} = \begin{pmatrix} 5 \times 7 & 5 \times 1 & 5 \times -3 \\ 5 \times 5 & 5 \times 3 & 5 \times -1 \end{pmatrix} = \begin{pmatrix} 35 & 5 & -15 \\ 25 & 15 & -5 \end{pmatrix}$$

$$\begin{aligned} 3A - 2B &= 3A + (-2B) \\ &= \begin{pmatrix} 3 & 6 \\ -18 & 9 \end{pmatrix} + \begin{pmatrix} -8 & 4 \\ -10 & -2 \end{pmatrix} \\ &= \begin{pmatrix} 3 - 8 & 6 + 4 \\ -18 - 10 & 9 - 2 \end{pmatrix} \\ &= \begin{pmatrix} -5 & 10 \\ -28 & 7 \end{pmatrix} \end{aligned}$$

1.5 Matrix Multiplication

When we multiply two matrices together, we will obtain a new matrix that may have a different order. There are two important points to be made here:

- Not every pair of matrices can be multiplied. $A \times B$ may not exist!
- Matrix multiplication is a **non-commutative** operation. This means that $A \times B$ is *not* equivalent to $B \times A$ and does not necessarily yield the same result.

So how can we tell if $A \times B$ exists, and what order it should have? If the number of columns of the first matrix equals the number of rows of the second matrix, then they can be multiplied, and the other dimensions tell us what kind of matrix we will obtain. In particular, if A is an $m_1 \times n_1$ matrix, and B is an $m_2 \times n_2$ matrix, then we can perform $A \times B$ if and only if $n_1 = m_2$, and the result will be a $m_1 \times n_2$ matrix (the result has the same number of rows as the first matrix and the same number of columns as the second matrix).

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

For example, if we multiply a 3×1 matrix by a 1×2 matrix, this will exist because the "1"s match, and the result will be a 3×2 matrix from the other dimensions.

To actually do the multiplication, we can then imagine lifting the rows of the first matrix and overlaying them with the columns of the second, the matched pairs of numbers are multiplied together and then added up. This is easiest to explain by an example:

$$\begin{aligned} \begin{pmatrix} 2 & 0 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & -2 \\ -4 & 5 \end{pmatrix} &= \begin{pmatrix} 2 \times 1 + 0 \times -4 & 2 \times -2 + 0 \times 5 \\ 1 \times 1 + 3 \times -4 & 1 \times -2 + 3 \times 5 \end{pmatrix} \\ &= \begin{pmatrix} 2 + 0 & -4 + 0 \\ 1 - 12 & -2 + 15 \end{pmatrix} \\ &= \begin{pmatrix} 2 & -4 \\ -11 & 13 \end{pmatrix} \end{aligned}$$

Example 1.6. *Let:*

$$A = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$$

Then:

$$AC = \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} = \begin{pmatrix} 1 \times -1 + 0 \times 4 & 1 \times 2 + 0 \times 5 \\ 2 \times -1 + -1 \times 4 & 2 \times 2 + -1 \times 5 \end{pmatrix} = \begin{pmatrix} -1 & 2 \\ -6 & -1 \end{pmatrix}$$

$$CA = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 2 & -1 \end{pmatrix} = \begin{pmatrix} -1 \times 1 + 2 \times 2 & -1 \times 0 + 2 \times -1 \\ 4 \times 1 + 5 \times 2 & 4 \times 0 + 5 \times -1 \end{pmatrix} = \begin{pmatrix} 3 & -2 \\ 14 & -5 \end{pmatrix}$$

In this example, $AC \neq CA$.

In general, the order of matrix multiplication *can not be changed.* In fact, one order might not even exist whilst the other does - as in the following example.

Example 1.7. *Let,*

$$B = \begin{pmatrix} 3 \\ -2 \end{pmatrix}, \qquad C = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix}$$

and calculate BC *and* CB *if they exist.*

First, as B *is a* 2×1 *and* C *is a* 2×2 *matrix,* BC *does not exist as the columns of* B *do not match the number of rows of* C*. However,* CB *does exist, and the result will be another* 2×1 *matrix:*

$$CB = \begin{pmatrix} -1 & 2 \\ 4 & 5 \end{pmatrix} \begin{pmatrix} 3 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \times 3 + 2 \times -2 \\ 4 \times 3 + 5 \times -2 \end{pmatrix} = \begin{pmatrix} -7 \\ -2 \end{pmatrix}$$

Example 1.8. *Given that:*

$$A = \begin{pmatrix} 1 & 2 \\ -6 & 3 \end{pmatrix}, \qquad B = \begin{pmatrix} 4 & -2 \\ 5 & 1 \end{pmatrix}$$

Calculate the following:

$$A \times B, \qquad B \times A$$

As A and B are both 2×2 matrices, both $A \times B$ and $B \times A$ will exist and will also be 2×2 matrices, similar to the example above.

Solution:

$$\begin{aligned} A \times B &= \begin{pmatrix} 1 & 2 \\ -6 & 3 \end{pmatrix} \begin{pmatrix} 4 & -2 \\ 5 & 1 \end{pmatrix} \\ &= \begin{pmatrix} 1 \times 4 + 2 \times 5 & 1 \times -2 + 2 \times 1 \\ -6 \times 4 + 3 \times 5 & (-6)(-2) + 3 \times 1 \end{pmatrix} \\ &= \begin{pmatrix} 4 + 10 & -2 + 2 \\ -24 + 15 & 12 + 3 \end{pmatrix} \\ &= \begin{pmatrix} 14 & 0 \\ -9 & 15 \end{pmatrix} \end{aligned}$$

$$
\begin{aligned}
B \times A &= \begin{pmatrix} 4 & -2 \\ 5 & 1 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -6 & 3 \end{pmatrix} \\
&= \begin{pmatrix} 4 \times 1 + (-2)(-6) & 4 \times 2 + (-2) \times 3 \\ 5 \times 1 + 1 \times (-6) & 5 \times 2 + 1 \times 3 \end{pmatrix} \\
&= \begin{pmatrix} 4 + 12 & 8 - 6 \\ 5 - 6 & 10 + 3 \end{pmatrix} \\
&= \begin{pmatrix} 16 & 2 \\ -1 & 13 \end{pmatrix}
\end{aligned}
$$

1.6 The Identity Matrix

For each positive integer n, the $n \times n$ **identity matrix** consists of one's on the diagonal entries and zeroes elsewhere. That is:

$$I = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \end{pmatrix}$$

and this is the only matrix which satisfies, for a matrix A of appropriate dimensions,

$$AI = IA = A$$

So the identity acts like a matrix version of the number "1" in the real numbers.

Consider the 2×2 and 3×3 identity matrices:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

1.7 Zero Matrix

This is a square matrix where every entry is zero. For example:

$$\underline{O} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{or} \quad \underline{O} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

It acts like the number 0 in matrix addition and matrix multiplication, so:

$$A\underline{O} = \underline{O} = \underline{O}A \quad \text{for any matrix } A \text{ of suitable order.}$$

1.8 Transpose

Let A be an $m \times n$ matrix. This means that A has m rows and n columns of entries. To obtain the transpose of a matrix A (denoted A^T), swap the rows and columns (or reflect all of the elements about the diagonal).

$$\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

Example 1.9.

$$\begin{pmatrix} 2 & -6 & 1 \\ 14 & 3 & -8 \end{pmatrix}^T = \begin{pmatrix} 2 & 14 \\ -6 & 3 \\ 1 & -8 \end{pmatrix}$$

2 Determinants

Square matrices (with dimensions $n \times n$) have a property called the **determinant**. This is a number (i.e. a scalar) associated with the matrix that is somewhat analogous to magnitude.

The determinant of matrix A can be denoted by $det(A)$ or $|A|$.

2.1 So what does the determinant *mean*?

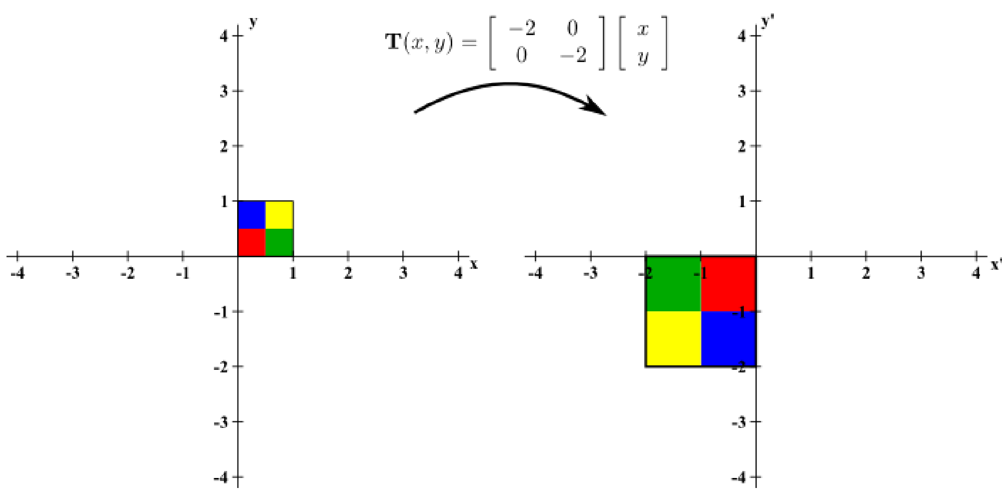
Previously we said matrix multiplication could transform an object in a 3d graphics engine.

If the matrix A encodes this action of rotating and stretching an object, then the determinant of A represents the **scaling factor**:

- If the absolute value is greater than 1, the matrix encodes an area-expanding transformation and the image is larger than the original object.
- If $|\det(A)| = 1$ the matrix is area-preserving.
- If the absolute value is less than 1 (that is $-1 < \det(A) < 1$) then the matrix encodes an area-contracting transformation that shrinks the image and pulls the vertices closer together.

A negative determinant further indicates that the orientation of the object is flipped (so it undergoes a reflection as well as rotation).

In this example, when every point in the square is multiplied by the matrix T with determinant 4, the transformed object is *four times as large* the original area:



2.2 Determinant of a 2×2 Matrix

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the determinant is very simple to calculate by multiplying the diagonal entries:

$$det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 2.1. *Given the square matrix*

$$A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$$

The determinant is given by:

$$det(A) = 3 \times 2 - (-1) \times 4 = 6 + 4 = 10$$

2.3 Determinant of a 3×3 Matrix

For a 3×3 matrix, select a row or column (any will suffice, but we usually use the top row) and multiply each of its entries by the determinant of the corresponding 2×2 co-matrix consisting of the rows and columns that the current entry is *not* in, and then also multiply by a positive or negative sign according to the checkerboard pattern:

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Therefore, given a 3×3 matrix $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$, choosing the top row:

$$\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a\begin{vmatrix} e & f \\ h & i \end{vmatrix} - b\begin{vmatrix} d & f \\ g & i \end{vmatrix} + c\begin{vmatrix} d & e \\ g & h \end{vmatrix} = a(ei - fh) - b(di - fg) + c(dh - eg)$$

Example 2.2.

Find the determinant of

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\begin{aligned} det(A) &= \begin{vmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix} \\ &= 3\begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} - (0)\begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ &= 3\big(0 \times 1 - (-2) \times 1\big) - 0 + 2\big(2 \times 1 - 0 \times 0\big) \\ &= 3(0 + 2) + 2(2 - 0) \\ &= 3 \times 2 + 2 \times 2 \\ &= 10 \end{aligned}$$

3 Inverse

If a square matrix A has *non-zero determinant*, then there exists a unique matrix A^{-1} with the same dimensions such that

$$AA^{-1} = A^{-1}A = I$$

this is the corresponding inverse matrix A^{-1}

If the determinant of a square matrix is equal to zero, then that matrix has no inverse. It is not "invertible".

3.1 Inverse of a 2×2 Matrix

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ with non-zero determinant $(ad - bc \neq 0)$, the inverse is given by:

$$A^{-1} = \frac{1}{det(A)} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

This may stated in formula booklets as:

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

where

$$|A| = \det(A) = ad - bc$$

Example 3.1.

Given the square matrix

$$A = \begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix}$$

The inverse exists as the determinant is:

$$\det(A) = (3)(2) - (-1)(4) = 6 + 4 = 10 \neq 0$$

Then the inverse itself is given by:

$$A^{-1} = \frac{1}{3 \times 2 - (-1) \times 4} \begin{pmatrix} 2 & -(-1) \\ -4 & 3 \end{pmatrix} = \frac{1}{6+4} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} = \frac{1}{10} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} = \begin{pmatrix} 1/5 & 1/10 \\ -2/5 & 3/10 \end{pmatrix}$$

We can check that we have obtained the correct answer by checking that $AA^{-1} = I$*:*

$$\begin{aligned} &\begin{pmatrix} 3 & -1 \\ 4 & 2 \end{pmatrix} \times \frac{1}{10} \begin{pmatrix} 2 & 1 \\ -4 & 3 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} 3 \times 2 + (-1) \times (-4) & 3 \times 1 + (-1) \times 3 \\ 4 \times 2 + 2 \times (-4) & 4 \times 1 + 2 \times 3 \end{pmatrix} \\ &= \frac{1}{10} \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \end{aligned}$$

and similarly
$A^{-1}A = I$.

Example 3.2. *For the following square matrices, find the inverse matrix if it exists.*

$$A = \begin{pmatrix} 1 & -1 \\ 0 & 2 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ -3 & 2 \end{pmatrix} \quad C = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

Solution:

All three of these matrices are 2×2 *and hence are square matrices (the first criteria for whether or not they are invertible).*

The determinant of A *is* $(1)(2)-(-1)(0) = 2 \neq 0$ *and hence it has an inverse, given by:*

$$A^{-1} = \frac{1}{(1)(2)-(-1)(0)} \begin{pmatrix} 2 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1/2 \\ 0 & 1/2 \end{pmatrix}$$

The determinant of B *is* $(1)(2)-(0)(-3) = 2 \neq 0$ *and hence it has an inverse, given by:*

$$B^{-1} = \frac{1}{(1)(2)-(0)(-3)} \begin{pmatrix} 2 & 0 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 3/2 & 1/2 \end{pmatrix}$$

However, the determinant of C *is* $(1)(-1)-(1)(-1) = 0$ *and hence it's inverse does not exist. It is a "singular" matrix.*

3.2 Inverse of a 3×3 Matrix

There are four basic steps to this method for determining the inverse of a 3×3 matrix (if you wish, you can use other methods such as Gaussian elimination):

1. Calculate the "matrix of minors".
2. Create the Co-factor Matrix.
3. Determine the Adjunct Matrix.
4. Finally, multiply the Adjunct Matrix by $1/Determinant$.

Example 3.3. *Find the inverse of*

$$A = \begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix}$$

1. Calculate the "matrix of minors":

To do this, for each element of the matrix: ignore the values on the current row and column, and calculate the determinant of the remaining values. Then put these determinants into a matrix.

$$\begin{pmatrix} \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 2 & -2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 2 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 1 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 0 & 1 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 0 & 1 \end{vmatrix} \\ \begin{vmatrix} 0 & 2 \\ 0 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 2 \\ 2 & -2 \end{vmatrix} & \begin{vmatrix} 3 & 0 \\ 2 & 0 \end{vmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \times 1 - (-2) \times 1 & 2 \times 1 - (-2) \times 0 & 2 \times 1 - 0 \times 0 \\ 0 \times 1 - 2 \times 1 & 3 \times 1 - 2 \times 0 & 3 \times 1 - 0 \times 0 \\ 0 \times (-2) - 2 \times 0 & 3 \times (-2) - 2 \times 2 & 3 \times 0 - 0 \times 2 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{pmatrix}$$

2. *Create the Co-factor Matrix:*

To turn the matrix of minors into the co-factor matrix, apply a checkerboard pattern of minus signs on alternate entries.

$$\begin{pmatrix} 2 & 2 & 2 \\ -2 & 3 & 3 \\ 0 & -10 & 0 \end{pmatrix} \quad \textit{Applying the sign-switching pattern:} \begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

$$\textit{Results in the co-factor matrix:} \begin{pmatrix} 2 & -2 & 2 \\ 2 & 3 & -3 \\ 0 & 10 & 0 \end{pmatrix}$$

3. *Determine the Adjunct Matrix (also known as the adjugate or adjoint matrix):*

The adjunct matrix is the transpose of the co-factor matrix, which means we reflect the matrix across the diagonal:

$$\begin{pmatrix} 2 & -2 & 2 \\ 2 & 3 & -3 \\ 0 & 10 & 0 \end{pmatrix}^T = \begin{pmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{pmatrix}$$

4. *Finally, multiply the Adjunct Matrix by* $1/Determinant$*:*

In Step 1, we already obtained most of the information required to calculate the determinant. Going across the top row of A *and multiplying each entry by the corresponding co-factor:*

$$det(A) = 3(2) + 0(2) + 2(2) = 6 + 4 = 10$$

Thus, the inverse of A *is given by:*

$$A^{-1} = \frac{1}{10} \begin{pmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{pmatrix} = \begin{pmatrix} 1/5 & 1/5 & 0 \\ -1/5 & 3/10 & 1 \\ 1/5 & -3/10 & 0 \end{pmatrix}$$

5. *We can check that we have obtained the correct answer by checking that* $AA^{-1} = I$*:*

$$\begin{pmatrix} 3 & 0 & 2 \\ 2 & 0 & -2 \\ 0 & 1 & 1 \end{pmatrix} \times \frac{1}{10} \begin{pmatrix} 2 & 2 & 0 \\ -2 & 3 & 10 \\ 2 & -3 & 0 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 3 \times 2 + 0 \times (-2) + 2 \times 2 & 3 \times 2 + 0 \times 3 + 2 \times (-3) & 3 \times 0 + 0 \times 10 + 2 \times 0 \\ 2 \times 2 + 0 \times (-2) + (-2) \times 2 & 2 \times 2 + 0 \times 3 + (-2) \times (-3) & 2 \times 0 + 0 \times 10 + (-2) \times 0 \\ 0 \times 2 + 1 \times (-2) + 1 \times 2 & 0 \times 2 + 1 \times 3 + 1 \times -3 & 0 \times 0 + 1 \times 10 + 1 \times 0 \end{pmatrix}$$

$$= \frac{1}{10} \begin{pmatrix} 10 & 0 & 0 \\ 0 & 10 & 0 \\ 0 & 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Similarly we could check that $A^{-1}A = I$*.*

4 Solving Simultaneous Equations using Matrices

You may have already met pairs of simultaneous linear equations and solved them by two methods: elimination and substitution. However, they can also be solved using an alternative matrix method.

4.1 Method

Given a pair of simultaneous equations:

$$ax + by = p$$

$$cx + dy = q$$

1. Write the pair of equations as a matrix equation

   $$AX = B$$

   and extract the matrix of coefficients $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

2. Calculate the inverse matrix A^{-1} of the matrix of coefficients.

3. Pre-multiply both sides by the inverse matrix to solve for the vector X:

   $$A^{-1}AX = A^{-1}B \implies X = A^{-1}B$$

4. From the entries in X, read off the values of x and y.

5. Substitute the values of x and y back into the original equations to verify solutions.

4.2 Example 1:

Solve for x and y,

$$5x + 2y = 10$$

$$4x - 3y = 14$$

Re-writing this as a matrix equation,

$$\begin{pmatrix} 5 & 2 \\ 4 & -3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

so we have $AX = B$, where

$$A = \begin{pmatrix} 5 & 2 \\ 4 & -3 \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}, \qquad B = \begin{pmatrix} 10 \\ 14 \end{pmatrix}$$

Then,

$$A^{-1} = \frac{1}{(5)(-3) - (2)(4)} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix} = \frac{-1}{23} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix}$$

and so

$$X = A^{-1}B = \frac{-1}{23} \begin{pmatrix} -3 & -2 \\ -4 & 5 \end{pmatrix} \begin{pmatrix} 10 \\ 14 \end{pmatrix} = \begin{pmatrix} 58/23 \\ -30/23 \end{pmatrix}$$

Thus we find $x = 58/23$ and $y = -30/23$.

4.3 Example 2:

Solve for x and y,

$$3x - 5y = 7$$

$$2x + 4y = 20$$

Re-writing this as a matrix equation,

$$\begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 20 \end{pmatrix}$$

so we have $AX = B$, where

$$A = \begin{pmatrix} 3 & -5 \\ 2 & 4 \end{pmatrix}, \quad X = \begin{pmatrix} x \\ y \end{pmatrix}, \quad B = \begin{pmatrix} 7 \\ 20 \end{pmatrix}$$

Then,

$$A^{-1} = \frac{1}{(3)(4) - (-5)(2)} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix} = \frac{1}{22} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix}$$

and so

$$X = A^{-1}B = \frac{1}{22} \begin{pmatrix} 4 & 5 \\ -2 & 3 \end{pmatrix} \begin{pmatrix} 7 \\ 20 \end{pmatrix} = \begin{pmatrix} 64/11 \\ 23/11 \end{pmatrix}$$

Thus we find $x = 64/11$ and $y = 23/11$.

5 Eigenvalues and Eigenvectors

5.1 Motivation

In computer aided design (CAD), a graphical model of a physical object can be manipulated by applying a linear transformation to the co-ordinates of each mesh point in a wire-frame diagram. This can be expressed in general matrix form as:

$$\underline{\mathbf{y}} = A\underline{\mathbf{x}}$$

where $\underline{\mathbf{x}}$ is the initial co-ordinate of a point, $\underline{\mathbf{y}}$ is the co-ordinate it gets mapped to after the manipulation, and A encodes the action of the transformation. We use matrix multiplication to determine $\underline{\mathbf{y}}$.

Example 5.1. *Consider a two-dimensional graphical model to which the following transformation matrix is applied:*

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}.$$

Under the action of this transformation, calculate what happens to points with various co-ordinates:

$$i)\ \underline{\boldsymbol{x}}_1 = \begin{pmatrix} 3 \\ -4 \end{pmatrix}, \quad ii)\ \underline{\boldsymbol{x}}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}, \quad iii)\ \underline{\boldsymbol{x}}_3 = \begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix}$$

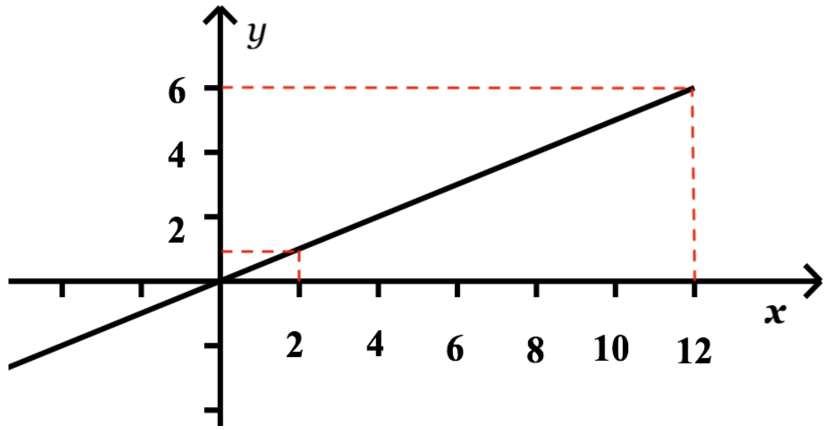
Solution:

$$i)\ \underline{\boldsymbol{y}}_1 = A\underline{\boldsymbol{x}}_1 = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 3 \\ -4 \end{pmatrix} = \begin{pmatrix} 15-8 \\ 6-8 \end{pmatrix} = \begin{pmatrix} 7 \\ -2 \end{pmatrix}.$$

$$ii)\ \underline{\boldsymbol{y}}_2 = A\underline{\boldsymbol{x}}_2 = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 10+2 \\ 4+2 \end{pmatrix} = \begin{pmatrix} 12 \\ 6 \end{pmatrix} = 6\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

$$iii)\ \underline{\boldsymbol{y}}_3 = A\underline{\boldsymbol{x}}_3 = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix} = \begin{pmatrix} 10\alpha+2\alpha \\ 4\alpha+2\alpha \end{pmatrix} = \begin{pmatrix} 12\alpha \\ 6\alpha \end{pmatrix} = 6\begin{pmatrix} 2\alpha \\ \alpha \end{pmatrix} = 6\alpha\begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

In the second and third cases, the output vector has the same direction as the input vector. In particular, any vector with the same direction as these will retain its direction *after the action of A (i.e. these are vectors for which the action of A does* not *rotate them). The magnitude of the vector may change, but the direction is preserved.*



Graphically, any point which lies on the line shown will be mapped to a point on the same line after the transformation. This line could be regarded as a "natural direction" or "natural axis" of the transformation.

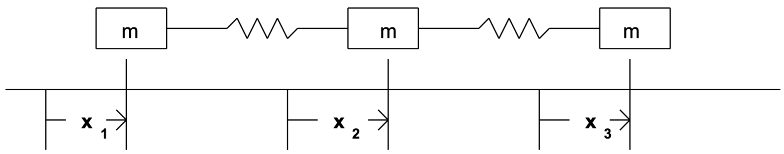
The question is, how many such axes are there for a general transformation matrix A, and how can we determine them systematically? Furthermore, for vectors that lie on these axes, how will their magnitude (for points, their distance from the origin) be affected by the action of A? This is the eigenvalue and eigenvector problem.

In particular, we want to find the scalar values λ (called **eigenvalues**) and associated column vectors $\underline{\mathbf{x}}$ (called **eigenvectors**) such that:

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}}$$

To see why these quantities are useful, we will now look at two different physical problems, where the solution can be found from the eigenvalues and eigenvectors of a matrix associated with the problem.

Example 5.2 (Harmonic Oscillators). *Consider three objects, each of mass m, coupled as shown by springs of stiffness (force constant) k. Let x_1, x_2, x_3 represent each object's displacement from equilibrium.*



It can be shown (by combining Hooke's law and Newton's Second Law) that the motion of these three objects can be modelled by the following set of second-order ordinary differential equations:

$$
\begin{aligned}
m\ddot{x}_1 &= k(x_2 - x_1) \\
m\ddot{x}_2 &= -k(x_2 - x_1) + k(x_3 - x_2) \\
m\ddot{x}_3 &= -k(x_3 - x_2)
\end{aligned}
$$

We want to represent this set of ODEs as a matrix problem. To see this, first rewrite the equations:

$$
\begin{aligned}
m\ddot{x}_1 &= -k\big(x_1 - x_2 + 0x_3\big) \\
m\ddot{x}_2 &= -k\big(-x_1 + 2x_2 - x_3\big) \\
m\ddot{x}_3 &= -k\big(0x_1 - x_2 + x_3\big)
\end{aligned}
$$

We can write these equations as the rows of matrices, and then separate the right-hand-side by constructing a matrix A of the coefficients of x_1, x_2, x_3:

$$
\begin{pmatrix} m\ddot{x}_1 \\ m\ddot{x}_2 \\ m\ddot{x}_3 \end{pmatrix} = \begin{pmatrix} -k\big(x_1 - x_2 + 0x_3\big) \\ -k\big(-x_1 + 2x_2 - x_3\big) \\ -k\big(0x_1 - x_2 + x_3\big) \end{pmatrix} = -k \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}
$$

Thus, we obtain the equivalent matrix equation for this problem:

$$
m\ddot{\underline{\boldsymbol{x}}} = -kA\underline{\boldsymbol{x}}, \quad \textit{where} \quad \underline{\boldsymbol{x}} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \ddot{\underline{\boldsymbol{x}}} = \begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{pmatrix} \quad \textit{and} \quad A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.
$$

Suppose we seek a solution of the form $\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}\cos(\omega t)$ *with constants* ω *and* $\underline{\boldsymbol{b}} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}$.
That is, $x_i = b_i \cos(\omega t)$ *for* $i = 1, 2, 3$*, so each of the masses oscillates with the same frequency but potentially with different amplitudes.*

Twice differentiating this equation for $\underline{\boldsymbol{x}}$*, we obtain:*

$$\dot{\underline{\boldsymbol{x}}} = -\omega \underline{\boldsymbol{b}}\sin(\omega t)$$

$$\ddot{\underline{\boldsymbol{x}}} = -\omega^2 \underline{\boldsymbol{b}}\cos(\omega t) = -\omega^2 \underline{\boldsymbol{x}}.$$

Using this relationship and the initial statement $m\ddot{\underline{\boldsymbol{x}}} = -kA\underline{\boldsymbol{x}}$*, we obtain:*

$$A\underline{\boldsymbol{x}} = \frac{-m}{k}\ddot{\underline{\boldsymbol{x}}} = \frac{m\omega^2}{k}\underline{\boldsymbol{x}},$$

and so

$$A\underline{\boldsymbol{b}}\cos(\omega t) = \frac{m\omega^2}{k}\underline{\boldsymbol{b}}\cos(\omega t)$$

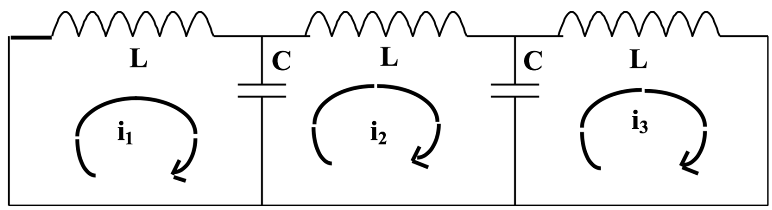
$$\therefore \quad A\underline{\boldsymbol{b}} = \left(\frac{m\omega^2}{k}\right)\underline{\boldsymbol{b}}, \quad \textit{where} \quad A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Therefore the values of $\lambda = \frac{m\omega^2}{k}$ *which satisfy this equation are the eigenvalues of matrix* A*. They give the frequencies of oscillatory (harmonic) motion, and the corresponding eigenvectors give the amplitudes of this motion* b_1, b_2, b_3*. So if the problem was to determine the possible harmonic frequencies of this vibrating system, they could be found by:*

$$\omega = \sqrt{\frac{\lambda k}{m}}, \qquad \textit{where } \lambda \textit{ are the eigenvalues of matrix } A.$$

Solving the eigenvalues and eigenvectors of A *will thus give us* $\underline{\boldsymbol{b}}$ *and* λ *from which we can obtain* ω*. Therefore we will gain both pieces of information required to fully specify the solution* $\underline{\boldsymbol{x}} = \underline{\boldsymbol{b}}\cos(\omega t)$ *and thus fully-understand how this system behaves.*

Example 5.3 (Circuits)**.** *Consider the electronic circuit shown.*



It has the following equations associated with it, whcih define the three electric currents:

$$L\frac{di_1}{dt} + \frac{1}{C}\int_0^t (i_1 - i_2)dt + E_1 = 0$$

$$\frac{1}{C}\int_0^t (i_2 - i_1)dt - E_1 + L\frac{di_2}{dt} + \frac{1}{C}\int_0^t (i_2 - i_3)dt + E_2 = 0$$

$$\frac{1}{C}\int_0^t (i_3 - i_2)dt - E_2 + L\frac{di_3}{dt} = 0$$

where E_1 and E_2 are the initial potentials on the capacitors.

Differentiating each of these equations with respect to t, we obtain:

$$L\frac{d^2 i_1}{dt^2} + \frac{1}{C}(i_2 - i_1) = 0$$

$$L\frac{d^2 i_2}{dt^2} + \frac{1}{C}\big((i_2 - i_1) - (i_2 - i_3)\big) = 0$$

$$L\frac{d^2 i_3}{dt^2} + \frac{1}{C}(i_3 - i_2) = 0$$

and this can be represented in matrix form by:

$$L\frac{d^2\underline{\boldsymbol{i}}}{dt^2} = -\frac{1}{C}A\underline{\boldsymbol{i}}, \quad \text{where} \quad \underline{\boldsymbol{i}} = \begin{pmatrix} i_1 \\ i_2 \\ i_3 \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

and L and C are scalar constants.

Letting $\underline{\boldsymbol{i}} = \underline{\boldsymbol{b}}\,\mathrm{e}^{j\omega t}$, we can obtain $A\underline{\boldsymbol{b}} = \lambda\underline{\boldsymbol{b}}$ where $\lambda = \omega^2 LC$. This situation is therefore mathematically very similar to the previous example, however the practical interpretation of the eigenvalues is different.

5.2 Method: Evaluation of Eigenvalues and Eigenvectors

Given a square matrix, we will now consider how we go about finding these eigenvalues and eigenvectors.

Consider a square matrix,

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

When this matrix acts on a column vector $\underline{\mathbf{x}}$, we obtain a new vector $A\underline{\mathbf{x}}$ that may be stretched and rotated in some way. We want to find the solutions to

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}},$$

where λ is a scalar, so that matrix multiplication by A preserves the direction of the vector $\underline{\mathbf{x}}$.

For a 2×2 matrix A, there are two such eigenvalues $\lambda = \lambda_1, \lambda_2$ and their associated eigenvectors $\underline{\mathbf{x}} = \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2$. Note that if $\underline{\mathbf{e}}_i$ is an eigenvector of A with eigenvalue λ_i, then so is *any scalar multiple* of $\underline{\mathbf{e}}_i$, so we can obtain a direction for the eigenvector, and then a vector in that direction of any magnitude will suffice.

First, we re-arrange the equation to obtain $(A - \lambda I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$, where I is the **identity matrix** that has 1's on the diagonal entries and 0's elsewhere,

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Then in order to find the non-trivial solutions (i.e. excluding $\underline{\mathbf{x}} = \underline{\mathbf{0}}$) we determine the eigenvalues by calculating the determinant of $A - \lambda I$ and solving the values of λ for which this is zero[1].

That is, we solve:

Characteristic equation of matrix A: $|A - \lambda I| = 0$

[1] This is because the existance of such a vector is the same as the matrix $A - \lambda I$ being "singular", or *not* invertible, which is equivalent to having determinant zero.

This will give us the **characteristic polynomial** (or characteristic equation) of A, and for a 2×2 matrix will be a quadratic equation. Solving this gives a pair of roots $\lambda = \lambda_1, \lambda_2$. For each of these, we can then obtain a corresponding non-zero eigenvector $\underline{\mathbf{x}} = \underline{\mathbf{e}}_1, \underline{\mathbf{e}}_2$ by solving:

$$A\underline{\mathbf{x}} = \lambda\underline{\mathbf{x}} \quad \text{or} \quad (A - \lambda I)\underline{\mathbf{x}} = \underline{\mathbf{0}}$$

for $\underline{\mathbf{x}}$.

5.3 Examples of calculating Eigenvalues and Eigenvectors

Example 5.4 (2×2 matrix)**.**

$$\textit{Let } A = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix},$$

then

$$A - \lambda I = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = \begin{pmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{pmatrix}.$$

Therefore, we wish to solve

$$|A - \lambda I| = 0 \quad \Longrightarrow \quad \begin{vmatrix} 1-\lambda & 2 \\ 3 & -4-\lambda \end{vmatrix} = 0.$$

Since this is a 2×2 matrix, we find the determinant by taking the difference of the product of the diagonals:

$$(1-\lambda)(-4-\lambda) - (2)(3) = 0,$$

and so

$$\lambda^2 + 3\lambda - 10 = 0 \qquad \textit{(The characteristic polynomial of matrix A.)}$$

Solving this quadratic equation yields two distinct, real, integer roots:

$$\lambda_1 = -5, \quad \lambda_2 = 2. \quad \textit{These are the eigenvalues of A.}$$

Next, we solve the eigenvectors one at a time. For the first eigenvalue, $\lambda_1 = -5$, let's call the corresponding eigenvector $\underline{\boldsymbol{e}}_1 = \begin{pmatrix} x \\ y \end{pmatrix}$.

To find the values of the components x and y, we need to solve:

$$A\underline{\boldsymbol{e}}_1 = -5\underline{\boldsymbol{e}}_1$$

which means

$$\begin{pmatrix} 1 & 2 \\ 3 & -4 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = -5 \begin{pmatrix} x \\ y \end{pmatrix}$$

This yields a pair of simultaneous equations:

$$\begin{aligned} x + 2y &= -5x \\ 3x - 4y &= -5y \end{aligned}$$

These are linearly dependent (i.e. they are the same equation, just rearranged in different ways), and solving either of them gives $y = -3x$*. If we choose* $x = 1$ *(and we can, since recall that* any *scalar multiple of the eigenvector will work, so the important property to preserve is the* relative *values of the two components), then we will get* $y = -3$*, and so one eigenvector corresponding to* $\lambda_1 = -5$ *is:*

$$\underline{\boldsymbol{e}}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Similarly, the second eigenvector (corresponding to eigenvalue $\lambda_2 = 2$*) is*

$$\underline{\boldsymbol{e}}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Check this as an exercise.

Example 5.5 (3×3 matrix)**.** *Consider the 3×3 matrix A that has appeared Examples 4.2 and 4.3. We will calculate the three eigenvalues and associated eigenvectors for this matrix.*

$$A = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Then $|A - \lambda I| = 0$ gives:

$$\begin{vmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{vmatrix} = 0$$

Recall the method of calculating determinants of 3×3 matrices. We select a row or column (any will suffice, but we usually use the top row) and multiply each of its entries by the determinant of the corresponding 2×2 co-matrix consisting of the rows and columns that the current entry is not *in, and then also multiply by a positive or negative sign according to the pattern:*

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}.$$

The resulting terms are summed to obtain the determinant.

Hence, in this case (using the top row) we have:

$$(1-\lambda)\begin{vmatrix} 2-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} - (-1)\begin{vmatrix} -1 & -1 \\ 0 & 1-\lambda \end{vmatrix} + 0\begin{vmatrix} -1 & 2-\lambda \\ 0 & -1 \end{vmatrix} = 0$$

$$(1-\lambda)\big((2-\lambda)(1-\lambda) - (-1)(-1)\big) + \big((-1)(1-\lambda) - (-1)(0)\big) = 0$$

$$(1-\lambda)\big((2-\lambda)(1-\lambda) - 1\big) - (1-\lambda) = 0$$

$$(1-\lambda)\big((2-\lambda)(1-\lambda) - 2\big) = 0$$

$$(1-\lambda)(\lambda^2 - 3\lambda + 2 - 2) = 0$$

$$(1-\lambda)(\lambda)(\lambda - 3) = 0$$

Hence there are three eigenvalues: $\lambda = 0, 1, 3$.

> ***Note:*** *In this case, we have successfully factorised by keeping out the common factor of* $(\lambda - 1)$. *In general, you are not expected to solve cubic equations. Therefore, you may be given the eigenvalues and asked to* ***verify*** *them. This means that you must obtain the characteristic polynomial, and then show that substituting in the proposed value of the eigenvalue* λ *satisfies the equation.*
>
> *e.g. If we had multiplied out the characteristic polynomial to obtain* $\lambda^3 - 4\lambda^2 + 3\lambda = 0$, *we could verify that* $\lambda = 3$ *is an eigenvalue in the following way:*
>
> $$(3)^3 - 4(3)^2 + 3(3) = 27 - 4 \times 9 + 9 = 27 - 36 + 9 = 0$$

Now we must obtain the eigenvectors which correspond to each of these. For a general value of λ *and a corresponding eigenvector* $\underline{\boldsymbol{x}}$, *the equation* $(A - \lambda I)\underline{\boldsymbol{x}} = \underline{\boldsymbol{0}}$ *gives:*

$$\begin{pmatrix} 1-\lambda & -1 & 0 \\ -1 & 2-\lambda & -1 \\ 0 & -1 & 1-\lambda \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

From the components of this we obtain the following three simultaneous equations:

$$\begin{aligned} (1-\lambda)x_1 - x_2 \qquad\qquad &= 0 \\ -x_1 + (2-\lambda)x_2 \quad -x_3 &= 0 \\ - x_2 \quad +(1-\lambda)x_3 &= 0 \end{aligned}$$

Hence,

i) For the first eigenvalue $\lambda_1 = 0$:

$$\begin{aligned} x_1 - x_2 \qquad &= 0 \quad (E_1) \\ -x_1 + 2x_2 - x_3 &= 0 \quad (E_2) \\ - x_2 + x_3 &= 0 \quad (E_3) \end{aligned}$$

From (E_1) *we have* $x_1 = x_2$*, and from* (E_3) *we obtain* $x_3 = x_2$*. This is all we need to do in this case, but we can check the consistency of the equations by using* $(E_4 = E_2 + E_1)$ *to eliminate* x_1*:*

$$x_2 - x_3 = 0 \quad (E_4)$$

and so we see that (E_4) *is just the same as* (E_3)*.*
Then we let $x_2 = \alpha$*, where* α *is just an arbitrary constant, and use the previous results to obtain both other co-ordinates in terms solely of* α*:*

$$x_1 = x_2 = \alpha, \quad \text{and} \quad x_3 = x_2 = \alpha.$$

Hence we have the eigenvector

$$\underline{\boldsymbol{x}}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

As before, this represents an infinite set of eigenvectors that all have the same direction but can be of any non-zero magnitude.

ii) $\lambda_2 = 1$*:*

$$\begin{aligned} -x_2 \quad &= 0 \quad (E_1) \\ -x_1 + x_2 - x_3 &= 0 \quad (E_2) \\ -x_2 \quad &= 0 \quad (E_3) \end{aligned}$$

Clearly (E_1) *and* (E_3) *are identical and give* $x_2 = 0$*.*
Substituting this result into (E_2) *then yields* $x_3 = -x_1$ *or* $x_1 = -x_3$*.*
Therefore let $x_1 = \beta$ *(an arbitrary constant) to obtain:*

$$\underline{\boldsymbol{x}}_2 = \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

iii) $\lambda_3 = 3$:

$$\begin{aligned} -2x_1 - x_2 \quad\quad &= 0 \quad (E_1) \\ -x_1 - x_2 - x_3 &= 0 \quad (E_2) \\ -x_2 - 2x_3 &= 0 \quad (E_3) \end{aligned}$$

Eliminate x_1 *from* (E_2) *using* $(E_4) = 2(E_2) - (E_1)$:

$$-x_2 - 2x_3 = 0 \quad (E_4)$$

and then eliminate x_2 *from* (E_3) *using* $(E_5) = (E_3) - (E_4)$. *As expected, this results in the tautology:*

$$0 = 0 \quad (E_5)$$

Then from (E_4): $x_2 = -2x_3$, *and from* (E_1): $x_1 = -\frac{1}{2}x_2 = x_3$. *Therefore let* $x_3 = \gamma$, *and we obtain* $x_2 = -2\gamma$ *and* $x_1 = \gamma$, *so that the eigenvector is:*

$$\underline{\boldsymbol{e}}_3 = \gamma \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

The complete solution to the problem is therefore:

$$\lambda_1 = 0,\ \underline{\boldsymbol{e}}_1 = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}; \qquad \lambda_2 = 1,\ \underline{\boldsymbol{e}}_2 = \beta \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}; \qquad \lambda_3 = 3,\ \underline{\boldsymbol{e}}_3 = \gamma \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$$

You can verify the solutions by calculating the matrix multiplications $A\underline{\boldsymbol{e}}_1$, $A\underline{\boldsymbol{e}}_2$, $A\underline{\boldsymbol{e}}_3$ *and checking that we get the product of the corresponding eigenvalue and eigenvector each time.*

5.4 Unit Vectors

It is sometimes useful to consider "normalised" or "unit" vectors. These have magnitude (size) equal to 1. To normalise a vector, we find its magnitude and then divide the vector by this scalar value. The direction is unchanged, so the components in each direction will still have the same ratio.

A unit vector is one that has magnitude equal to one. Given any vector, $\underline{\mathbf{v}}$ we can find a unit vector in the same direction by:

$$\hat{\underline{\mathbf{v}}} = \frac{\underline{\mathbf{v}}}{|\underline{\mathbf{v}}|}$$

Example 5.6. *Consider the eigenvectors we found in Example 4.4. The first eigenvector is:*

$$\underline{\boldsymbol{e}}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$$

Therefore it has magnitude:

$$|\underline{\boldsymbol{e}}_1| = \sqrt{(1)^2 + (-3)^2} = \sqrt{1+9} = \sqrt{10}$$

So a unit vector in the same direction as $\underline{\mathbf{e}}_1$, which we denote by $\hat{\underline{\mathbf{e}}}_1$ is:

$$\hat{\underline{\boldsymbol{e}}}_1 = \frac{\underline{\boldsymbol{e}}_1}{|\underline{\boldsymbol{e}}_1|} = \frac{1}{\sqrt{(1)^2 + (-3)^2}} \begin{pmatrix} 1 \\ -3 \end{pmatrix} = \frac{1}{\sqrt{10}} \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Similarly, the second normalised eigenvector is

$$\hat{\underline{\boldsymbol{e}}}_2 = \frac{\underline{\boldsymbol{e}}_2}{|\underline{\boldsymbol{e}}_2|} = \frac{1}{\sqrt{(2)^2 + (1)^2}} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

5.5 General Properties of Eigenvalues

Theorem 5.1. *The determinant of an invertible square matrix is equal to the product of it's eigenvalues. That is, for an invertible $n \times n$ matrix A with eigenvalues $\lambda_1, \dots, \lambda_n$:*

$$|A| = \lambda_1 \lambda_2 \dots \lambda_n$$

This is because the characteristic polynomial of A can be factorised in the following way:

$$|A - \lambda I| = (\lambda_1 - \lambda)(\lambda_2 - \lambda) \dots (\lambda_n - \lambda)$$

and so setting $\lambda = 0$ results in $|A| = \lambda_1 \lambda_2 \dots \lambda_n$.

Theorem 5.2. *The sum of the eigenvalues of a square matrix is equal to the "trace" of the matrix, that is, the sum of its diagonal elements.*

$$tr(A) = \lambda_1 + \lambda_2 + \dots + \lambda_n$$

These two theorems can be used to obtain two eigenvalues of an $n \times n$ matrix A from a pair of simultaneous equations, given that the other $(n - 2)$ are already known.

There are many other properties of eigenvalues and eigenvectors for different classes of matrices, but they are not within the scope of this course. Matrix algebra is fundamental to one of the largest and most active areas of pure mathematics, as well as its many physical applications.

5.6 Application: Linear stability analysis of systems of ODEs

Eigenvalues are an extremely important concept in algebra, and have many uses when related to matrices that model physical problems.

One important application is in **stability analysis** for certain kinds of dynamical systems.

If we have a system of **linear** ODEs of the form:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = ax_1 + bx_2 + cx_3$$

$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = dx_1 + ex_2 + fx_3$$

$$\frac{\mathrm{d}x_3}{\mathrm{d}t} = gx_1 + hx_2 + ix_3$$

we can write it in matrix form as:

$$\dot{X} = AX$$

Where:

$$X = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \qquad \dot{X} = \begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{pmatrix}, \qquad \text{and} \qquad A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$$

We say that A is the **Jacobian matrix** for this dynamical system.

(The dot above a variable denotes the derivative w.r.t. time)

Such a system has an **equilibrium** at the origin (where $x_1 = x_2 = x_3 = 0$), as this means that:

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = \frac{\mathrm{d}x_2}{\mathrm{d}t} = \frac{\mathrm{d}x_3}{\mathrm{d}t} = 0$$

so every variable has zero rate of change with respect to time - i.e. the system is static. More complex non-linear systems may have other equilibria, or lack one at the origin, but we will only deal with this case.

A common question in the design of control systems, is when is this equilibrium **stable**?

Stability means that if the system is disturbed from the equilibrium by an arbitrarily small amount, it will tend to return (small perturbations dissipate). In an unstable system, the disturbance is amplified and over time the system will move away from the equilibrium. If this is desirable depends on the context - consider chemical reactions in an industrial process, some you want to sustain but others could be dangerous.

Returning to eigenvalues then. . .

Stability criterion:

The equilibrium of such a linear ODE system is:

- *Stable* if all of the eigenvalues of the Jacobian matrix have negative real part.
- *Unstable* otherwise.

5.6.1 Example

Consider a process governed by the differential equations:

$$\dot{x} = x - y$$

$$\dot{y} = -x + 2y - z$$

$$\dot{z} = -y + z$$

Writing this in matrix form:

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{z} \end{pmatrix} = \begin{pmatrix} 1x - 1y + 0z \\ -1x + 2y - 1z \\ 0x - 1y + 1z \end{pmatrix} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

we can see that this system has Jacobian equal to matrix A that we have studied previously.

We check that there exists an equilibrium at the origin. If $x = y = z = 0$, then clearly from the original set of equations:

$$\dot{x} = \dot{y} = \dot{z} = 0 \quad \text{hence, equilibrium.}$$

(However, note that in this system, this zero-equilibrium is actually part of a family of equilibria for *any* case where $x = y = z$. Can you see why this is the case?)

Now, earlier we found that A had eigenvalues $\lambda = 0, 1, 3$. Thus, according to the criterion, the presence of positive, real eigenvalues means that this equilibrium is **unstable** and the system will tend to be repelled from it.

6 MATLAB and Matrices

6.1 Declaring matrices

To name and store a matrix in Matlab, use square brackets and write the list of elements in each row from left to right, starting with the top row. Separate the elements in each row by a space, and separate the rows themeselves with a semicolon.

To declare the following matrix:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$$

we write:

```
A = [1 2 3; 4 5 6];
```

6.2 Matrix algebra

Addition, subtraction, scalar multiplication and matrix multiplication all work very simply in the way you would expect.

```
A = [1 2; 3 -4];

B = [4 -6; -1 2]

C = A + B          (matrix addition)

D = A - B          (matrix subtraction)

E =  5*A          (scalar multiplication)

F = A*B          (matrix multiplication)
```

Don't forget that the order of matrix multiplication is very important!

6.3 Determinant, Transpose and Inverse

Simple commands exist for most important matrix operations.

```
A = [1 2; 3 -4];

B = transpose(A);          (matrix B is the transpose of A)

C = inv(A);          (C is the inverse matrix of A)

d = det(A);          (obtain the determinant of A)
```

Remember that the inverse does not exist for a non-square matrix. If you attempt this, you will receive the error:

```
Error using inv. Matrix must be square.
```

and the script will fail. However, you must also remember that the inverse of a square matrix does not exist if the determinant is zero, and you should always check this before attempting to calculate the inverse. If you ask Matlab for the inverse of a matrix with zero determinant it *will not fail*, and you will simply receive the message:

```
Warning: Matrix is singular to working precision.
```

It will be up to you to realise that this means you should not proceed.

6.4 Eigenvalues and Eigenvectors

MATLAB can automatically determine the eigenvalue and eigenvector pairs of a square matrix for you, using the Symbolic Math Toolbox:

```
A = [1 2; 3 -4];

B = sym(A);

[vecA,valA] = eig(B);
```

This first creates a symbolic version B of the matrix (which Matlab needs to do in order to handle more abstract mathematical operations) and then produces two matrices, $valA$ contains the eigenvalues on the diagonal (with zeros elsewhere) and $vecA$ contains the eigenvectors in each column. The ordering between the two corresponds, so the first column of $vecA$ is the eigenvector corresponding to the eigenvalue in the first diagonal entry of $valA$.

To find a unit vector, we need to first obtain the magnitude of the vector, and then divide the vector by this magnitude:

```
X = [3; 17];          (declare a vector X)

mag = norm(X);          (obtain the magnitude of X and store it as mag)

unitX = X/mag;          (obtain the unit vector and store it as unitX)
```

6.5 Solving simultaneous equations

To use this method does not require any additional commands, but a combination of what we have learned. Let's revisit Example 1 from Section 4.2 and carry out the method in Matlab:

$$5x + 2y = 10$$

$$4x - 3y = 14$$

```
A = [5 2; 4 -3];

B = [10; 14];

X = inv(A)*B
```

Note that this will provide decimal answers. To obtain the precise fractions that we found when solving by hand, we simply ask Matlab to convert the answer to a symbolic variable:

```
sym(X)
```