Probability

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Contents

Today we will cover. . .

- Different types of probability.
- Drawing Venn diagrams to describe probabilities.
- Mutually exclusive events.
- Conditional probability, dependent and independent events.
- Probability trees.

Probability of an event

There are two types of probability:

A-Priori: We can make an exact calculation based on what we know about the system. For example, we know the probability of rolling a six on a fair die is $1/6$.

A-Posteriori: The probability can only be estimated from empirical data on similar past events. For example, of 126 matches in the FA cup between premier league and non-league sides, premier league teams won 102 times. So, the probability of a premier league side beating a non-league side next is $102/126$.

A-Priori events

If we have a specific result in mind, let's call it event A.

For example, the probability of rolling a 6 when we roll a die.

We can find the probability of this event A (denoted $P(A)$) by considering the number of possible outcomes where A occurs as a fraction of *all possible outcomes*:

$$P(A) = \frac{\text{No. of outcomes where A occurs}}{\text{No. of possible outcomes}}$$

A-Priori example

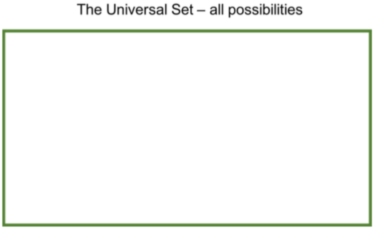
A card is drawn at random from a standard deck of 52 playing cards. What is the probability that it is a face card (i.e. a jack, queen or king)?

Let's call the event of picking a face card, A. How many cards do we have?
4 kings
4 queens
4 jacks
So in total we have 12 out of a possible 52 outcomes. Thus:

$$P(A) = \frac{12}{52} = \frac{3}{13} = 0.231 \text{ (3 d.p.)}$$

Venn diagrams

Venn diagrams are used to visualise sets of outcomes. Consider again the 52 cards. If we pick one card, then we have 52 possibilities in total - the **universal set** of all outcomes.



(a) Universal set

(b) $P(A)$

We then called the event of drawing an ace A, with probability $P(A)$. This is a subset of the universal set.

Venn diagrams

What about the possibility of getting a spade? Call this event S with probability, $P(S)$.

There are 13 spades, thus:

$$P(S) = \frac{13}{52} = \frac{1}{4}$$

Venn diagram showing the probability of drawing a spade (S):

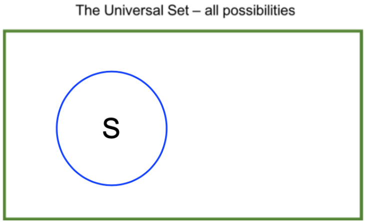


Figure: $P(S)$

“Or” probability

For two events A and B, the probability that A **or** B **or** both occur is given by:

“Or” probability of two generic events

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

If the two events A and B are **mutually exclusive**, they cannot both occur simultaneously. Thus $P(A \text{ and } B) = 0$ and so:

“Or” probability of two mutually exclusive events

$$P(A \text{ or } B) = P(A) + P(B)$$

Example: Mutually exclusive events

Draw a Venn diagram representing the probability of drawing a spade (S) or a diamond (D).

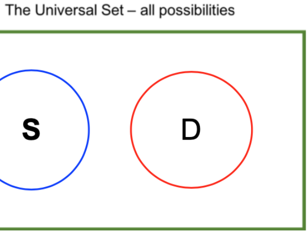


Figure: $P(S \text{ or } D)$

$$P(S \text{ or } D) = P(S) + P(D)$$
$$= \frac{13}{52} + \frac{13}{52} = \frac{26}{52} = \frac{1}{2}$$

Example: Overlapping events

Daw a Venn diagram showing the probability of drawing either a face card (Z) or a spade (S):

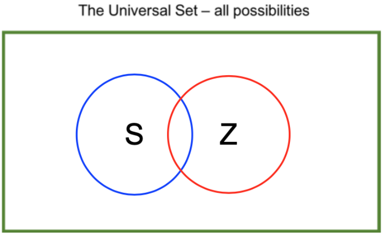


Figure: $P(Z \text{ or } S)$

Example: Overlapping events

A card can be **both** a spade and a face card, so we have to subtract the probability that it is both, as this is counted twice.

In the Venn diagram, this is interpreted as an overlap as they are occurring at the same time, so to work out the area we have to subtract the overlapping part once.

$$\begin{aligned} P(Z \text{ or } S) &= P(Z) + P(S) - P(Z \text{ and } S) \\ &= \frac{3}{13} + \frac{13}{52} - \frac{3}{52} \\ &= \frac{22}{52} = \frac{11}{26} = 0.423 \text{ (3 d.p.)} \end{aligned}$$

Conditional probability

Two events A and B are **dependent** if the occurrence of A affects the probability of B occuring. In this case, we can talk about the probability of B **given that** A **has already occurred**:

$$P(B|A)$$

Then the probability that *both* B and A occur is:

"And" probability of dependent events

$$P(A \text{ and } B) = P(A)P(B|A)$$

If the occurrence of A does *not* affect B, they are **independent**:

"And" probability of independent events

$$P(A \text{ and } B) = P(A)P(B)$$

Conditional probability

How can we calculate a conditional probability if the events *are* dependent?

In the case of $P(B|A)$, we can think of the "pool" of outcomes that we are interested in as being reduced to only those where A has already occurred. So for what fraction of *those* events did B occur?

This can be expressed as:

$$P(B|A) = \frac{P(A \text{ and } B)}{P(A)}$$

Example: Conditional probability

One card is drawn at random from a deck. Given that it is a red card R, what is the probability that it is specifically a diamond D?

There are 26 red cards (hearts and diamonds), so:

$$P(R) = \frac{26}{52}$$

and there are 13 diamonds, all of which are red. Hence:

$$P(D \text{ and } R) = \frac{13}{52}$$

Thus:

$$P(D|R) = \frac{P(D \text{ and } R)}{P(R)} = \frac{13/52}{26/52} = \frac{1}{2}$$

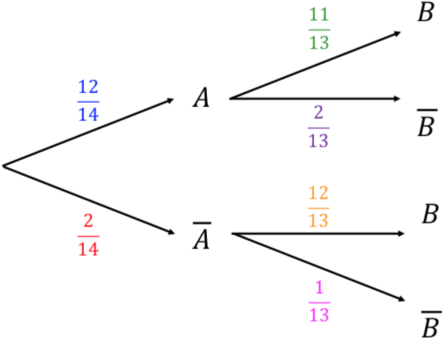
Probability trees

Probability trees provide us with a systematic means of describing more complicated problems that involve a sequence of choices with conditional probabilities.

Probability trees

A box contains 14 light bulbs: 12 are working and 2 are broken. Two bulbs are chosen at random without replacement. Call the outcome that the first bulb works A and the second bulb works B.

Construct a probability tree to account for all possible outcomes:



$$P(A \text{ and } B) = \frac{12\times 11}{14\times 13}$$

$$P(A \text{ and } \overline{B}) = \frac{12\times 2}{14\times 13}$$

$$P(\overline{A} \text{ and } B) = \frac{2\times 12}{14\times 13}$$

$$P(\overline{A} \text{ and } \overline{B}) = \frac{2\times 1}{14\times 13}$$

Probability trees

What is the probability that **both** bulbs work?

$$P(A \text{ and } B) = \frac{12}{14} \times \frac{11}{13} = \frac{132}{182} = 0.725 \text{ (3 d.p.)}$$

What is the probability that exactly **one** bulb works?

$$\begin{aligned} P(A \text{ and } \bar{B}) + P(\bar{A} \text{ and } B) &= \frac{12}{14} \times \frac{2}{13} + \frac{2}{14} \times \frac{12}{13} \\ &= \frac{48}{182} = \frac{24}{91} = 0.264 \text{ (3 d.p.)} \end{aligned}$$

Summary

If we are unsure what equation to use. . .